Dynamical Systems

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Dynamical Systems are systems, described by one or more equations, that evolve over time. For example, the growth of a population can be described by dynamic equations. Time can be understood to be either discrete (day 1, day 2 etc.) or continuous (3.4567... seconds). If we take time to be continuous, dynamical systems will be described by differential equations - equations that involve the derivative (the instantaneous change) of a function. If we take time to be discrete, dynamical systems will be described by difference equations - equations relating the value of a variable at time $t+1$ to its value at time t. We will look at both cases below.

1 Differential Equations

A **differential equation** is an equation which involves an unknown function $f(t)$ and at least one of its derivatives. Let $y = f(t)$. Then we denote $f'(t)$ as $\frac{df}{dt}(t)$ or as $\dot{y}(t)$. A general differential equation is then of the form

$$\dot{y} = F(y(t), t)$$

The purpose of this equation is not to solve for the variable t, but rather to solve for the function $y(t)$. (Since the function is the unknown, we use y rather than f to label it).

In Economics, differential equations are often used to express changes over time. For example, the change in the capital stock at time t, $\dot{K}(t)$, is a function of the current capital stock $K(t)$, the saving rate s, and the depreciation rate δ:

$$\dot{K}(t) = (s - \delta)K(t)$$

Therefore it is common to use t instead of x as the argument of f and refer to $\dot{f}$ or $\dot{y}$ as the change in f over time.

1.1 Types of Differential Equations

1. **1st order -** Equations which involve the first derivative $f'(t)$ of the function but no higher derivatives. For example:

$$\dot{y} = ky$$

   is a first order difference equation since it only involves one derivative of $f(t)$.

2. **Nth order -** Equations which involve the nth order derivatives $f^{(n)}(t)$ of the function. For example:

$$\ddot{y} = ky$$

   is a second order differential equation since it involves two derivatives of $f(t)$. Also

$$y^{(n)} = ky$$

   is an nth order differential equation.

3. **Ordinary -** An ordinary differential equation is a differential equation where the unknown function takes only one argument. For example, the differential equations mentioned thus far have all been ordinary. However, the equation

   $$y^{(n)} = ky(x,t) \cdot t$$

   is not ordinary since there are two arguments x and t, while

   $$y^{(n)} = ky(t) \cdot t$$

   is ordinary (the only variable is t). All the differential equations we will look at will be ordinary.

4. **Linear -** A differential equation is linear if it can be written in the form

   $$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y = a_0(t)$$

5. **Autonomous -** An autonomous differential equation is one where the only occasion when t enters the equation is through y. For example

   $$\dot{y} = ky$$

   is autonomous (t enters only through $y(t)$) while

   $$\dot{y} = ky \cdot t$$

   is not (t enters by itself, outside of $y(t)$).

1.2 Solving Linear Differential Equations

Differential equations are generally difficult to solve. Therefore, in this section of the course we will examine only first order linear difference equations:

$$\dot{y} + p(t)y = q(t).$$

1.2.1 Special Cases

Notice that if $p(t) = 0$, then we have a simple integration problem we all know how to solve

$$\dot{y} = q(t) \Rightarrow y = Q(t) + C,$$

where $Q(t)$ is the antiderivative of $q(t)$.
Now let $q(t) = 0$, and $p(t) = -k \in \mathbb{R}$, where k is any constant. Then we must solve the equation

$$\dot{y} = ky \Rightarrow \frac{dy}{dt} = ky(t).$$

Rewrite the equation by dividing both sides by $y(t)$. (We are assuming that $y(t) \neq 0$. $y(t) = 0$ is obviously a solution as well.). Then we have

$$\frac{\dot{y}}{y} = k.$$

Integrating both sides with respect to t we have

$$\ln(y) = kt + C \Rightarrow y = e^{kt+C} = \gamma e^{kt},$$

where $\gamma = e^C$.

1.2.2 General Case

If we allow $p(t)$ to be some function but leave $q(t) = 0$ we can follow the same steps as in the example with $p(t) = -k$ above and get

$$\ln(y) = H(t) + C \Rightarrow y = e^{H(t)+C} = \gamma e^{H(t)},$$

where $H(t) = \int p(t)dt$ and $\gamma = e^C$.

In the general case of

$$\dot{y} + p(t)y = q(t),\ \ p(t) \neq 0,\ \ q(t) \neq 0$$

is solved in the following manner. Define a function $H(t)$ such that

$$H(t) = \int p(t)dt.$$

Now multiply the above differential equation to get

$$\dot{y}e^{H(t)} + p(t)ye^{H(t)} = q(t)e^{H(t)}$$

Notice that by the chain rule, that

$$\frac{d}{dt}\left(ye^{H(t)}\right) = \dot{y}e^{H(t)} + p(t)ye^{H(t)}$$

which is equal to the left hand side of the differential equation. Therefore

$$\frac{d}{dt}\left(ye^{H(t)}\right) = q(t)e^{H(t)}.$$

We now integrate both sides to get

$$ye^{H(t)} = \int q(t)e^{H(t)}dt + C,$$

and then multiply through by $e^{-H(t)}$ to get the general form of the solution:

$$y = e^{-H(t)}\left\{\int q(t)e^{H(t)}dt + C\right\}.$$

Example Find the solution of the equation $\dot{y} = ay + b$.
In this case $p(t) = -a$ and $q(t) = b$. Therefore

$$H(t) = \int -adt = -at + C.$$

Plugging this into our general solution equation, we have that

$$y = e^{at-C}\left\{\int be^{-at+C}dt + D\right\}$$

$$y = e^{at-C}\left\{-\frac{b}{a}e^{-at+C} + D\right\}$$

$$y = -\frac{b}{a} + De^{at+C}$$

$$y = -\frac{b}{a} + \alpha e^{at},$$

where $\alpha = De^C$.

1.2.3 Additional Conditions

Sometimes in a problem we are given an initial condition or a terminal condition. We can use these conditions to help us find what the unknown scalars are in our solutions.

Examples

1. Let $y(0) = y_0$. Solve for the general solution to the problem $\dot{y} = ky$.
   We already know the general solution to this problem is
   $$y = \gamma e^{kt}.$$
   We know that when $x = 0$, $y = y_0$. Plugging these values into the equation we have
   $$y_0 = \gamma.$$
   Therefore, the general solution to the equation is $y = y_0 e^{kt}$.

2. Let $y(T) = y_T$. Solve for the general solution to the problem $\dot{y} = ky$.
   We already know the general solution to this problem is
   $$y = \gamma e^{kt}.$$
   We know that when $t = T$, $y = y_T$. Plugging these values into the equation we have
   $$y_T = \gamma e^{kT} \Rightarrow \gamma = y_T e^{-kT}$$
   Therefore, the general solution to the equation is $y = y_T e^{-kT} e^{kt} = y_T e^{k(t-T)}$.

1.3 Steady States and Phase Diagrams

1.3.1 Finding Steady States

A constant solution $y(t) \equiv c$ is called a **steady state** for a differential equation. It is a solution where the value of y does not change over time. For example, consider the equation for the capital stock given above
$$\dot{K}(t) = (s - \delta)K(t)$$
We are either saving more that the depreciation of the capital stock, less that the value of the depreciation of the capital stock, or just equal. If we are equal, then we will have the same amount of capital next period as we do this period. Then if the saving rate is the same next year as it is this year, we will just cover depreciation with nothing left over. The capital stock will remain the same as this year in the third year, and so on. This is called the steady state level of capital.

Now let us consider the differential equation $\dot{y} = ay$. In order for the level of y to be the same this year and last year, we must have that y does not change, or $\dot{y} = 0$. Therefore, the only value of y for which this can happen (as long as $a \neq 0$) is $y = 0$, and so $y = 0$ is a steady state to the equation.

In general, one finds the steady states to the differential equation $\dot{y} = F(y, t)$ by setting $\dot{y} = F(y, t) = 0$ and solving the resulting equation for y.

Example Find the steady state for the equation $\dot{y} = b + ay$.
Let $\dot{y} = 0$. Then $ay = -b$, and the steady state value of the solution is $y = -\frac{b}{a}$.

1.3.2 Phase Diagrams

A **phase diagram** is a way to illustrate the steady states of a homogenous differential equation and the behavior of solutions around the steady states. It is a graph of the differential equation $\dot{y}(t) = F(y(t))$ with the value of the function y on the horizontal axis and the change in y, $\dot{y}$, on the vertical axis. (Note that we cannot draw a one-dimensional phase diagram for a non-autonomous differential equations since in that case $\dot{y}$ changes with $y(t)$ and with t.)

Any point at which the graph intersects the horizontal axis, that is, at which $\dot{y} = 0$, is a steady state. At any point at which the graph of $\dot{y} = F(y)$ is above the horizontal axis, $\dot{y}$ is positive and therefore y is increasing. At any point at which the graph of $\dot{y} = F(y)$ is below the horizontal axis, $\dot{y}$ is negative and therefore y is decreasing. We can add arrows along the horizontal axis that indicate this behavior of y: The arrows will point to the right on any segment on which the graph is above the axis and to the left on any segment that is below the axis.

For example, let us consider the simple differential equation

$$\dot{y} = ay.$$

Here there are two cases:

Case one: $a > 0$**.** In this case, the curve is a linear function sloping upward. The only steady state is at $y = 0$. Notice that if $a > 0$, then when $y > 0$, $\dot{y} > 0$. This implies that y is growing over time. If $y < 0$, then $\dot{y} < 0$, and y is shrinking over time. Therefore, we can see that if the equation starts at any point other than $y_0 = 0$, the system will diverge to negative infinity or positive infinity. Another way to see this is to take the solution to the equation, $y = y_0 e^{at}$, and let $t \to \infty$. If $y_0 > 0$, then $y \to \infty$, and if $y_0 < 0$, then $y \to -\infty$.

Case two: $a < 0$**.** In this case, the curve is a linear function sloping downward. Again, the only steady state is at $y = 0$. Notice that if $a < 0$, then when $y > 0$, $\dot{y} < 0$. This implies that y is shrinking over time. If $y < 0$, then $\dot{y} > 0$, and y is growing over time. Therefore, we can see that if the equation starts at any point, it will eventually converge to $y = 0$. Another way to see this is to take the solution to the equation, $y = y_0 e^{at}$, and let $t \to \infty$. If $y_0 > 0$, then $y \to 0$, and if $y_0 < 0$, then $y \to 0$ also (remember, $a < 0$).

1.3.3 Stability

We call a steady state y^* in the domain **asymptotically stable** if $\exists\ r > 0$ and $B(y^*, r) \subset$ domain such that if we have as an initial point any $y \in B(y^*, r)$, the system will converge to y^* over time. We call a system stable if all points in the domain converge to a steady state. Notice that in the previous example, the system was stable when $a < 0$, but unstable when $a > 0$.

It is easy to tell the stability of a steady state from the phase diagram: If the arrows point towards the steady state from both sides, it is stable. Else it is unstable. A simple test for stability is as follows: Let $\dot{y} = F(y)$.

$$\text{If } \frac{dF(y)}{dy}|_{y^*} < 0, \text{ then the steady state } y^* \text{ is stable}$$

$$\text{if } \frac{dF(y)}{dy}|_{y^*} > 0, \text{ then the steady state } y^* \text{ is unstable}$$

Example Let $\dot{y} = y^2 - 4y + 3$. Determine the steady states and their stability.

Solution: Let $\dot{y} = 0$. Then $0 = y^2 - 4y + 3 = (y - 1)(y - 3)$. Therefore, we have two steady states, $y = 1$ and $y = 3$.

Next, take the derivative of $\dot{y} = y^2 - 4y + 3$ with respect to y to get

$$\frac{d\dot{y}}{dy} = 2y - 4.$$

Evaluated at $y = 1$, we have

$$\frac{d\dot{y}}{dy}|_{y^*} = 2(1) - 4 = -2 < 0,$$

therefore $y = 1$ is a stable steady state. Evaluated at $y = 3$, we have

$$\frac{d\dot{y}}{dy}|_{y^*} = 2(3) - 4 = 2 > 0,$$

and we have that $y = 3$ is an unstable steady state.

2 Systems of Differential Equations

Consider the general two-equation system of differential equations:

$$\dot{x}(t) = F(x(t), y(t), t)$$

$$\dot{y}(t) = G(x(t), y(t), t)$$

These look like two single differential equations, but the problem is that y appears in the equation for $\dot{x}$ and x appears in the equation for $\dot{y}$. We need to solve them simultaneously.

2.1 Steady States

Just as before, we can find the steady state of the system by setting both $\dot{x} = 0$ and $\dot{y} = 0$.

Examples

1. Let $\dot{x} = e^{x-1} - 1$ and $\dot{y} = ye^x$.
   Setting both these equations equal to 0 yields

$$\dot{x} = 0 \Rightarrow e^{x-1} = 1 \Rightarrow x = 1$$

$$\dot{y} = 0 \Rightarrow ye = 0 \Rightarrow y = 0$$

2. Let $\dot{x} = x + 2y$ and $\dot{y} = x^2 + y$.
   Setting both these equations equal to 0 yields

$$\dot{x} = 0 \Rightarrow x = -2y$$

$$\dot{y} = 0 \Rightarrow y = -x^2 \Rightarrow$$

$$x = -2\left(-x^2\right) \Rightarrow x(1 - 2x) = 0 \Rightarrow x = \{0, \frac{1}{2}\} \Rightarrow y = \{0, -\frac{1}{4}\}$$

   Therefore, the two steady states are $(x, y) = \left\{(0,0), \left(\frac{1}{2}, -\frac{1}{4}\right)\right\}$.

3. Let $\dot{x} = e^{1-x} - 1$ and $\dot{y} = (2 - y)e^x$.
   Setting both these equations equal to 0 yields

$$\dot{x} = 0 \Rightarrow e^{1-x} = 1 \Rightarrow x = 1$$

$$\dot{y} = 0 \Rightarrow (2 - y)e = 0 \Rightarrow y = 2$$

Phase Diagrams If we have a two dimensional system as in the example above, we can draw phase diagram with x on one axis and y on the other (unlike in the one dimensional case, we no longer plot $\dot{x}$ or $\dot{y}$ on an axis). We then draw in the curve in x, y - space along which $\dot{x} = 0$ and the curve along which $\dot{y} = 0$. The points of intersection of these two curves are the steady states of the system. To investigate the behavior of the system, find the signs of $\dot{x}$ and $\dot{y}$ in each of the segments of the plane divided by the $\dot{x} = 0$ and $\dot{y} = 0$ curves. For example, if $\dot{x} > 0$ and $\dot{y} < 0$, the system is increasing in x-direction and decreasing in y-direction over time.

2.2 Stability

For a single differential equation $\dot{y} = f(y)$, we could test whether the steady state was stable by checking whether

$$\left.\frac{df(y)}{dy}\right|_{y^*} < 0.$$

If so, then the differential equation was stable. The condition for systems of differential equations is more complicated, and deals with the eigenvalues of the Jacobian matrix of the system.

Let J be the Jacobian matrix of the system of differential equations. Then we can test for stability as follows:

- y^* is stable if and only if all eigenvalues of $J(y^*)$ are negative or have negative real parts.
- y^* is unstable if and only if some eigenvalue of $J(y^*)$ is positive or has positive real parts.

If the Jacobian at y^* has some pure imaginary or zero eigenvalues and no positive eigenvalues, then we cannot determine the stability of the steady state by looking at the Jacobian.

Examples revisited

1. Let $\dot{x} = e^{x-1} - 1$ and $\dot{y} = ye^x$.
   We already calculated that the steady state of the system will be $\mathbf{z} = (x, y) = (1, 0)$. The Jacobian of the system is
   $$\begin{pmatrix} e^{x-1} & 0 \\ ye^x & e^x \end{pmatrix}(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix},$$
   which implies that the eigenvalues of the system are 1 and e. Since both of these are positive, we have an unstable steady state.

2. Let $\dot{x} = x + 2y$ and $\dot{y} = x^2 + y$.
   We already calculated that the steady states of the system are $\mathbf{z} = (x, y) = \{(0, 0), (\frac{1}{2}, -\frac{1}{4})\}$. The Jacobian of the system is
   $$\begin{pmatrix} 1 & 2 \\ 2x & 1 \end{pmatrix}.$$
   When $\mathbf{z} = (0, 0)$, then we have the Jacobian
   $$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
   which implies that the repeated eigenvalue is 1. Since both of these are positive, $(0, 0)$ is an unstable steady state.
   When $\mathbf{z} = (\frac{1}{2}, -\frac{1}{4})$, then we have the Jacobian
   $$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

and solving for the eigenvalues yields $\lambda = 1 \pm \sqrt{2}$. Since one of them is positive, $(\frac{1}{2}, -\frac{1}{4}$ is also an unstable steady state.

3. Let $\dot{x} = e^{1-x} - 1$ and $\dot{y} = (2-y)e^x$.
   We already calculated that the steady state of the system will be $\mathbf{z} = (x, y) = (1, 2)$. The Jacobian of the system is
   $$\begin{pmatrix} -e^{1-x} & 0 \\ (2-y)e^x & -e^x \end{pmatrix}(\mathbf{z}) = \begin{pmatrix} -1 & 0 \\ 0 & -e \end{pmatrix},$$
   which implies that the eigenvalues of the system are -1 and $-e$. Since both of these are negative, we have a stable steady state.

2.3 Solving Systems of Linear Differential Equations

Consider the linear system of differential equations:

$$\dot{x} = a_{11}x + a_{12}y$$
$$\dot{y} = a_{21}x + a_{22}y$$

which can be expressed as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where $\mathbf{x} = (x, y)'$, $\dot{\mathbf{x}} = (\dot{x}, \dot{y})'$, and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Also assume that y_0 and x_0 are given.
Consider the case where A is a diagonal matrix, i.e that $a_{12} = a_{21} = 0$. Then the new system is

$$\dot{x} = a_{11}x,$$
$$\dot{y} = a_{22}y,$$

whose solution is

$$x = x_0 e^{a_{11}t},$$
$$y = y_0 e^{a_{22}t}.$$

That was easy! We can also easily see that the eigenvalues of the Jacobian matrix will be a_{11} and a_{22}, and therefore the system will be stable if both a_{11} and a_{22} are less than zero.
For the case where $a_{12} \neq 0$ or $a_{21} \neq 0$, the solution is more complicated. However, if we can diagonalize A, we can transform the system $\dot{\mathbf{x}} = A\mathbf{x}$ into $\dot{\mathbf{x}} = P\Lambda P^{-1}\mathbf{x}$, then multiply both sides by P to get

$$P^{-1}\dot{\mathbf{x}} = \Lambda P^{-1}\mathbf{x}.$$

If we define $\dot{\mathbf{w}} = P^{-1}\dot{\mathbf{x}}$ and $\mathbf{w} = P^{-1}\mathbf{x}$, then we have

$$\dot{\mathbf{w}} = \Lambda\mathbf{w}.$$

where Λ is a diagonal matrix. The solution for this system is

$$\mathbf{w}(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A and the diagonal entries of Λ, as we saw before.
Now we can transform the solution for $\mathbf{w}$ back to $\mathbf{x}$ by multiplying with P on the left:

$$\mathbf{x}(t) = P\mathbf{w}(t) = c_1 e^{\lambda_1 t}\mathbf{p_1} + c_2 e^{\lambda_2 t}\mathbf{p_2}$$

where $\mathbf{p_1}$ and $\mathbf{p_1}$ are the eigenvectors of A.

Example Solve the following system of differential equations:

$$\dot{x} = x - y$$

$$\dot{y} = -4x + y$$

The system can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation for A is

$$(1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0,$$

and therefore the eigenvalues of the matrix A are $\lambda = \{3, -1\}$. Therefore, we know the system will be unstable.

The matrix $A - I\lambda$ associated with $\lambda = 3$ is

$$\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix},$$

which implies that $y = -2x$, and therefore $(1, -2)'$ is the corresponding eigenvector. For $\lambda = 1$, we have that

$$A - I\lambda = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix},$$

which implies that $y = 2x$, or that $(1, 2)'$ is the corresponding eigenvector.

We can now form the matrix

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = P^{-1}\mathbf{x}$. We now have the system

$$\dot{\mathbf{w}} = \Lambda\mathbf{w},$$

whose solution is

$$w_x = c_1 e^{3t},$$

$$w_y = c_2 e^{-t}$$

where c_1 and c_2 are constants. Finally, since we have $\mathbf{w} = P^{-1}\mathbf{x}$, then $\mathbf{x} = P\mathbf{w}$,

$$x = w_x + w_y = c_1 e^{3t} + c_2 e^{-t}$$

$$y = -2w_x + 2w_y = -2c_1 e^{3t} + 2c_2 e^{-t}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

If we are given initial conditions $\mathbf{x}(0) = \mathbf{x}_0 = (x_0, y_0)^T$, we can find the values of the constants $\mathbf{c} = (c_0, c_1)^T$ from $\mathbf{c} = P^{-1}\mathbf{x}_0$. (This works because $\mathbf{c}$ are the initial conditions when our system is transformed to the variables $\mathbf{w} = P\mathbf{x}$.)

So in our example,

$$c_1 = w_{x0} = \frac{1}{2}x_0 - \frac{1}{4}y_0$$

$$c_2 = w_{y0} = \frac{1}{2}x_0 + \frac{1}{4}y_0$$

and

$$x = \left\{ \frac{1}{2}x_0 - \frac{1}{4}y_0 \right\} e^{3t} + \left\{ \frac{1}{2}x_0 + \frac{1}{4}y_0 \right\} e^{-t}$$

$$y = \left\{ -x_0 + \frac{1}{2}y_0 \right\} e^{3t} + \left\{ x_0 + \frac{1}{2}y_0 \right\} e^{-t}.$$

Notice that this procedure only works if A is diagonalizable. If A is not diagonalizable, we have to use something called the generalized eigenvectors of A and the formula for the general solution becomes slightly more complicated.
In particular, let A be a 2 x 2 matrix with one repeated eigenvalue λ and only one linearly independent eigenvector $\mathbf{p}$. Then there exists a vector $\mathbf{q}$ such that $(A - \lambda I)\mathbf{q} = \mathbf{p}$ and $(A - \lambda I)^2\mathbf{q} = \mathbf{0}$, which is called the generalized eigenvector of A.
The general solution to the system $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{p} + c_2\mathbf{q}) + te^{\lambda t}c_2\mathbf{p}.$$

(Why this works, see Simon & Blume, page 681.)

2.4 Non-Linear Systems

Finding general solutions of non-linear systems can be extremely difficult if not impossible. However, we can find a first-order estimate of the solution about a steady state using the Taylor rule. For example, assume our general system of equations:

$$\dot{x} = F(x, y)$$

$$\dot{y} = G(x, y)$$

Setting these equations equal to zero, we can solve for some steady state (x^*, y^*).
The Taylor expansion gives us an approximation of the function $h(x, y)$ around some point (x^*, y^*). Notice that the Taylor expansion in this case is

$$h(x, y) \approx h(x^*, y^*) + h_x(x, y)(x - x^*) + h_y(x, y)(y - y^*).$$

This is a linear approximation of a nonlinear function about (x^*, y^*).
We can use the Taylor approximation to rewrite the system of differential equations about (x^*, y^*):

$$\dot{x} \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$

$$\dot{y} \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*).$$

This can we rewritten is the form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{c}$, where

$$A = \begin{pmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{pmatrix}, \ \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and c_1, c_2 are constants. In a way, however, the constants don't matter because they just shift our phase diagram around. They don't actually affect the stability or motion of the system. We can then proceed to find an approximation of the system using the diagonalization process presented in the last section.

3 Difference Equations

A **difference equation** is an equation which evolves over *discrete* time intervals. For example, a difference equation would be a function which would tell us what the value of some variable y is for any given time t. We generally have some starting initial point for t (usually $t = 0$), and we may or may not have a terminal point for t. Consider the function

$$y_t = b^t y_0,$$

where $b \in \mathbb{R}$. This is a difference equation. We can plug whatever we want in for t and get the value of the function at that point. For example, if $b = 0.5$ and $y_0 = 1$, then we have the following evolution of y over time:

$$y_0 = 1,\ y_1 = 0.5,\ y_2 = .25, \ldots, y_n = \frac{1}{2^n}$$

Sometimes we are not given an explicit solution for y_t, but are given a rule of how y_t evolves over one period. For example,

$$y_{t+1} = (1 + b)y_t.$$

3.1 Solving Difference Equations

Say we have some initial condition y_0, and consider the difference equation

$$y_{t+1} = ay_t.$$

The equation can be solved iteratively by simply starting with $t = 0$, calculating y_1, then using that solution to calculate y_2, and so forth. For example:

$$y_0 = y_0$$

$$y_1 = ay_0$$

$$y_2 = ay_1 = a \cdot ay_0 = a^2 y_0$$

$$y_3 = ay_3 = a \cdot a^2 y_0 = a^3 y_0$$

$$\vdots$$

We can look at the pattern and infer that a solution to the difference equation will be of the form:

$$y_t = a^t y_0.$$

3.2 Relationship Between Continuous and Discrete Time

Subtract y_t from both sides of the equation

$$y_{t+1} = (1 + b)y_t$$

to get

$$\Delta y_t = y_{t+1} - y_t = by_t$$

Notice that this looks a lot like the differential equation

$$\dot{y} = by.$$

In fact, if instead of time moving in discrete units, time moved continuously, then the difference equation above would be the same as the differential equation. To see this, consider the difference equation governing the growth of an asset given a constant interest rate, and is compounded once per period:

$$A_{t+1} = A_t(1 + r).$$

Now consider the same equation, except time now is measured in half-increments. In other words, the interest is compounded twice per period, but at the same annual rate r:

$$A_{t+1} = A_t \left(1 + \frac{r}{2}\right)^2$$

Now consider the case where interest is compounded n times per period:

$$A_{t+1} = A_t \left(1 + \frac{r}{n}\right)^n.$$

The limit of this function as $n \to \infty$ is

$$A_{t+1} = A_t e^r,$$

Solving this difference equation we have

$$A_t = A_0 e^{rt}$$

which is the solution to the differential equation $\dot{A} = rA$. Therefore, the equation $A_{t+1} = (1 + r)A_t$ is the discrete time counterpart of the differential equation $\dot{A} = rA$.

3.3 Properties of the Solution $y_t = a^t y_0$

We have solved the differential equation $y_{t+1} = ay_t$. What happens to y_t as $t \to \infty$? There are seven cases, with subcases under them:

Case 1: $a > 1$. Here we can see that as $t \to \infty$, $a^t \to \infty$. Therefore, the system diverges to $+\infty$ or $-\infty$, depending on whether y_0 is greater or less than 0.

Case 2: $a = 1$. If $a = 1$, then the system will stay at its initial condition y_0, no matter what y_0 is.

Case 3: $a \in (0, 1)$. In this case, as $t \to \infty$, $a^t \to 0$. Therefore, $y_t \to 0$ as $t \to 0$. Also, the sequence of y_ts will be strictly monotonically decreasing if $y_0 > 0$, and monotonically increasing if $y_0 < 0$.

Case 4: $a = 0$. The system immediately jumps to $y_1 = 0$ no matter where y_0 is, and then $y_t = 0 \; \forall \; t$ thereafter.

Case 5: $a \in (-1, 0)$. In this case, as $t \to \infty$, $a^t \to 0$. Therefore, $y_t \to 0$ as $t \to 0$. However, the sequence of y_ts will oscillate between positive and negative values as $t \to \infty$.

Case 6: $a = -1$. If $a = -1$, then $y_t = y_0$ for $t \in \{\text{even integers}\}$ $y_t = -y_0$ for $t \in \{\text{odd integers}\}$.

Case 7: $a < -1$. As $t \to \infty$, the subsequence $a^{t_e} \to \infty$ and $a^{t_o} \to -\infty$, where t_e denotes even integers and t_o denotes odd integers. Therefore, the system oscillates between positive and negative values, with the magnitude of oscillate growing to infinity over time.

Subcase to all 7 cases: $y_0 = 0$. In the case $y_0 = 0$, we have that $y_t = 0 \; \forall \; t$.

3.4 Steady States

The condition for a steady state for differential equations was that $\dot{y} = 0$. For difference equations, the condition is similar:

$$\Delta y_t = 0, \text{ i. e.} y_{t+1} = y_t.$$

Intuitively, if we can find a value of y such which will be the same next period, then that level of y must be a steady state. Therefore, in our difference equation we plug in y_t for y_{t+1} and solve:

$$y_t = ay_t \Rightarrow y_t(1-a) = 0 \Rightarrow y_t = 0 \text{ or } a = 1$$

Therefore, the steady state solution to our differential equation is $y = 0$ for any value of a. This makes sense, because we already saw during our analysis of cases that $y_t = 0 \; \forall \; t$ when $y_0 = 0$. If $a = 1$, then any y is a steady state since $y_t = y_t$ for all y_t.

3.5 Stability of Steady States

Looking at our seven cases for the asymptotics of $y_{t+1} = ay_t$, we see that we converge to the steady state $y = 0$ whenever $|a| < 1$. We can generalize this condition by noticing that $a = \frac{\partial y_{t+1}}{\partial y_t}$. In general, a steady state y^* of a difference equation is

$$\text{unstable if } \left| \frac{\partial y_{t+1}}{\partial y_t}(y^*) \right| > 1$$

$$\text{stable if } \left| \frac{\partial y_{t+1}}{\partial y_t}(y^*) \right| < 1$$

This is very similar to the conditions for stability of differential equations.

3.6 Nonlinear example

Consider the following evolution of capital in an economy:

$$k_{t+1} = k_t - (\delta + n)k_t + sf(k_t)$$

This says that the level of capital per capita next year is equal to the level of capital this year, minus depreciated capital and capital dilution from population growth, plus the difference between output and consumption. We seek to analyze the dynamics of capital in this economy. Assume that $f(k_t)$ is a concave function, and $f(0) = 0$, $f'(0) = \infty$, and $\lim_{n\to\infty} f'(n) = 0$. Also assume that $\delta + n < 1$, $\delta, n, s \in (0, 1)$.

First we wish to find a steady state level of capital per capita. We set $k_{t+1} = k_t$ to get

$$k_{t+1} = k_t - (\delta + n)k_t + sf(k_t) \Rightarrow$$

$$(\delta + n)k^* = sf(k^*).$$

This will pin down some level of capital k^*. Notice that $k^* = 0$ is a steady state, as will some $k^* > 0$ by the concavity of $f(k)$. We can find this point graphically by plotting k_{t+1} as a function of k_t and determining its point of intersection with the 45 degree line, along which $k_t = k_{t+1}$.

Checking for stability, we take the derivative of the function k_{t+1} with respect to k_t to get

$$\frac{\partial k_{t+1}}{\partial k_t} = 1 - (\delta + n) + sf'(k_t)$$

Therefore, we have that the system is stable if $f'(k^*) < \frac{\delta+n}{s}$. For the steady state $k^* = 0$, we have that $f'(0) = \infty$, and therefore the system is unstable in this case. However, by the concavity of $f(k_t)$ the steady state $k^* > 0$ must be stable since $\frac{\delta+n}{s}k^* = f(k^*)$. Therefore we have one stable steady state and one unstable steady state.

4 Homework

4.1 Single Differential Equations

1. Find the general solutions to the following equations.

   (a) $\dot{y} - 2y = 1$

   (b) $2\dot{y} + 5y = 2$

   (c) $\dot{y} - 2y = 1 - 2x$

   (d) $x\dot{y} - 4y = -2nx$

   (e) $\dot{y} = e^x y$

2. Find the general solutions to the previous questions given that $y_0 = 1$.

3. Draw the phase diagram for the equation $\dot{k}(t) = f(k) - c - (\delta + n)k$, where $f(k) > 0$ when $k > 0$, $f(k) = 0$ when $k = 0$, and $f(k)$ is a concave function which intersects the line $c + (d + n)k = f(k)$ at two points. Also draw the phase diagram for $\frac{\dot{c}(t)}{c} = \frac{1}{\sigma}(f'(k) - \theta - n)$, where $\sigma, \theta, n \in \mathbb{R}$, and $f(k)$ is the same as before. Check the stability of each of the equations. Find their general solutions. Graph k and c is separate graphs with respect to time on the horizontal axis.

4.2 Difference Equations

Solve the difference equation $y_{t+1} = by_t^2$. Find the steady states. Graph the function with respect to time. Draw the phase diagrams. Determine stability of the steady states. (Hint: there will be several cases.) What about $by_{t+1} = ay_t^2 + c$? (This is complicated, so think about it graphically for starters. Don't worry if you can't finish the whole problem with all the cases.)